

# Boundary Value Problems for Nabla Fractional Difference Equations of order less than one

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**Abstract**— In this Paper, a fractional order ( $0 < \alpha < 1$ ) nabla difference equation satisfying two point boundary value conditions is considered. Existence and uniqueness of solutions are established using Contraction mapping theorem and nonlinear contraction mapping theorem. The results are illustrated by some examples.this column.

**Index Terms**— Fractional Difference equations; Boundary value Problems; Existence of solutions; Contaction mapping theorem.

**AMS Classification**— 39A10, 39A99.

## 1 INTRODUCTION

The theory of difference equations [2, 3, 13, 15] has gained importance due to two important reasons viz.,

a). Their applicability in different areas of science, engineering and technology involving discrete phenomenon

b). Discretization of differential equations to suit the needs of digital era.

Though many of the results in the theory of difference equations are generalization of the corresponding results in the theory of ordinary difference equations.

Studies about discrete boundary value problems already exist in the literature ([5, 6, 15, 16] and references therein).

Fractional calculus [11] gained the attention of many mathematicians and engineers due to its applicability in diverse fields such as visco-elasticity, control theory, neurology etc. But the analogous theory of discrete fractional calculus was initiated recently and some inequalities,

comparison principles and solutions of fractional difference equations were developed [1, 9, 10, 12, 14].

Moreover, a good number of papers dealing with discrete fractional boundary value problems of order ( $\alpha > 1$ ) and involving forward difference operator ( $\Delta$ ) are available in the literature [7, 8]. But much not been yet done for ( $\alpha < 1$ ). Recently, Rue A.C .Ferreira has initiated the study in that direction [17] involving difference operator.

With the motivation derived from the above analysis, in this

paper we establish existence and uniqueness of solutions to

two point boundary value problem (TPBVP) associated with fractional difference equations of order  $\alpha$  ( $0 < \alpha < 1$ ) involving backward difference operator ( $\nabla$ ).

This paper is organised as follows: in Section 2, we give some definitions and fundamental results from the theory of discrete fractional calculus that are necessary for our study. Section 3 contains main results and in Section 4, the results established are illustrated with examples.

**2 PRELEMANIES**

Throught out this paper, for notations and terminology we refer [15]. For any real number  $n_0$ , denote  $N_{n_0}^+ = \{n_0, n_0+1, \dots\}$ . For any functuion  $u : N_{n_0}^+ \rightarrow R$ , backward difference operator or nabla difference operator is defined as  $\nabla u(n) = u(n)-u(n-1)$ .

Definition2.1.

The extended binomial coefficient  $\binom{a}{n}$  ( $a \in R, n \in Z$ ) is

$$\text{defined as } \binom{a}{n} = \begin{cases} \frac{\Gamma(a+1)}{\Gamma(a-n+1)\Gamma(n+1)} & n > 0 \\ 1 & n = 0 \\ 0 & n < 0 \end{cases}$$

(2.1)

Definition2.2. For any complex number  $\alpha$  and  $\beta$ ,

$(\alpha)_\beta$  is defined as

$$= \begin{cases} \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)}, & \alpha, \beta \text{ \& } \alpha + \beta \text{ are neither 0 nor negative} \\ & \text{int eger} \\ 1, & \alpha = \beta = 0 \\ 0, & \alpha = 0 \text{ \& } \beta \text{ is neither 0 nor negative int eger} \\ \text{undefined,} & \text{otherwise} \end{cases} \quad (2.2)$$

Remark1.

For any complex number  $\alpha$  and  $\beta$ , when  $\alpha, \beta$  and  $\alpha + \beta$  are neither zero nor negative integers, then for any positive integer,

$$(\alpha + \beta)_n = \sum_{k=0}^n \binom{n}{k} (\alpha)_{(n-k)} (\beta)_k.$$

Though fractional difference and sum operators have been defined in different ways, the definitions given by Hirota [14] and Atsushi Nagai [1] (which is a slight modification of Hirota's) using the extended binomial coefficient are very interesting

Definition2.3. [1] Let  $u(n) : N_0^+ \rightarrow R, m-1 \leq \alpha \leq m, \alpha \in R^+$  and  $m \in N_1^+$ . Then the fractional sum operator of order  $\alpha$  is defined as

$$\begin{aligned} \nabla^{-\alpha} u(n) &= \sum_{j=0}^{n-1} \binom{j+\alpha-1}{j} u(n-j) \\ &= \sum_{j=1}^{n-1} \binom{n-j+\alpha-1}{n-j} u(j) \end{aligned}$$

(2.3)

Definition2.4. The Caputo type fractional difference operator of order  $\alpha$  is defined as

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$$\nabla^\alpha u(n) = \sum_{j=0}^{n-1} \binom{j-\alpha+m-1}{j} \nabla^m u(n-j) \tag{2.4}$$

Corollary1. [12] The equivalent form of (2.4) is given by

$$\nabla^\alpha u(n) = \sum_{j=1}^{n-1} \binom{n-j-\alpha-1}{n-j} u(j) - \sum_{k=0}^{m-1} \binom{n+k-\alpha-1}{n-1} [\nabla^k u(j)]_{j=0}. \tag{2.5}$$

Proof. We prove the statement (2.5) by using mathematical induction on m.

For m=1, (2.4) becomes

$$\begin{aligned} \nabla^\alpha u(n) &= \sum_{j=0}^{n-1} \binom{j-\alpha}{j} \nabla u(n-j) \\ &= u(n) - \binom{n-1-\alpha}{n-1} u(0) \\ &\quad - \alpha \sum_{j=1}^{n-1} \frac{1}{(j-\alpha)} \binom{j-\alpha}{j} u(n-j) \\ &= - \binom{n-1-\alpha}{n-1} u(0) + \sum_{j=0}^{n-1} \binom{j-\alpha-1}{j} u(n-j) \\ \nabla^\alpha u(n) &= - \binom{n-1-\alpha}{n-1} u(0) + \sum_{j=1}^n \binom{n-j-\alpha-1}{n-j} u(j) \end{aligned}$$

.For m=2, (2.4) becomes

$$\begin{aligned} \nabla^\alpha u(n) &= \sum_{j=0}^{n-1} \binom{j-\alpha+1}{j} \nabla^2 u(n-j) \\ &= \sum_{j=0}^{n-1} \binom{j-\alpha+1}{j} \nabla[\nabla u(n-j)] \\ &= \sum_{j=0}^{n-1} \binom{j-\alpha}{j} \nabla u(n-j) - \binom{n-\alpha}{n-1} [\nabla u(n-j)]_{j=n} \\ &= \sum_{j=0}^{n-1} \binom{j-\alpha-1}{j} u(n-j) - \binom{n-1-\alpha}{n-1} u(0) \\ &\quad - \binom{n-\alpha}{n-1} [\nabla u(n-j)]_{j=n} \\ &= \sum_{j=1}^{n-1} \binom{n-j-\alpha-1}{n-j} u(j) - \sum_{k=0}^{m-1} \binom{n+k-\alpha-1}{n-1} [\nabla^k u(j)]_{j=0}. \end{aligned}$$

The Statement is true for m=2.

We assume that the statement is true for m-1. Then

$$\begin{aligned} \nabla^\alpha u(n) &= \sum_{j=0}^{n-1} \binom{n-j-\alpha-1}{n-j} u(j) \\ &\quad - \sum_{k=0}^{m-2} \binom{n+k-\alpha-1}{n-1} [\nabla^k u(j)]_{j=0}. \end{aligned}$$

Now we prove that the statement is true for m. Consider

$$\begin{aligned} \nabla^\alpha u(n) &= \sum_{j=0}^{n-1} \binom{j-\alpha+m-1}{j} \nabla^m u(n-j) \\ &= \sum_{j=0}^{n-1} \binom{j-\alpha+m-2}{j} \nabla^{m-1} u(n-j) \\ &\quad - \binom{n-\alpha+m-2}{n-1} [\nabla^{m-1} u(n-j)]_{j=n} \\ &= \sum_{j=1}^{n-1} \binom{n-j-\alpha-1}{n-j} u(j) - \sum_{k=0}^{m-1} \binom{n+k-\alpha-1}{n-1} [\nabla^k u(j)]_{j=0}. \end{aligned}$$

Hence by principle of mathematical induction, the statement is true for m.

Definition2.5. The Reimann-Liouville type fractional difference operator of order  $\alpha$  is defined as

$$\nabla^\alpha u(n) = \nabla^m \left[ \sum_{j=1}^m \binom{n-j+m-\alpha-1}{n-j} u(j) \right]. \tag{2.6}$$

Corollary2. The equalent form of (2.6) is given by

$$\nabla^\alpha u(n) = \sum_{j=1}^n \binom{n-j-\alpha-1}{n-j} u(j). \tag{2.7}$$

Proof: Consider

$$\nabla^\alpha u(n) = \nabla^m \left[ \sum_{j=1}^m \binom{n-j+m-\alpha-1}{n-j} u(j) \right].$$

$$= \nabla^{m-1} \nabla \left[ \sum_{j=1}^n \binom{n-j+m-\alpha-1}{n-j} u(j) \right]$$

$$= \nabla^{m-1} \left[ \sum_{j=1}^n \binom{n-j+m-\alpha-1}{n-j} u(j) - \sum_{j=1}^{n-1} \binom{n-j+m-\alpha-2}{n-j-1} u(j) \right]$$

$$= \nabla^{m-1} \left[ \sum_{j=1}^n \binom{n-j+m-\alpha-2}{n-j-1} u(j) \right].$$

Applying the procedure m-1 times we get the required result.

Remark2. Let  $u(n)$  be any function defined for  $n \in N_0^+$  and  $f(n, r)$  be a function defined on  $n \in N_0^+, 0 \leq \alpha \leq \infty$ . Then for

$n \geq 0$  and  $0 < \alpha < 1$ , if

$\nabla^\alpha u(n) = f(n, u(n))$  then

$$u(n) = u(0) + \sum_{j=0}^{n-1} D(n-1, \alpha, j) f(j, u(j)) \tag{2.8}$$

Where  $D(n-1, \alpha, j) = \binom{n-j+\alpha-2}{n-j-1}$ .

### 3 MAIN RESULTS

In this section we consider the following the following boundary value problem associated with fractional difference equation of order  $0 < \alpha < 1$ ,

$$\nabla^\alpha u(n) = f(n, u(n)) \tag{3.1}$$

$$au(0)+bu(N+1) = 0. \tag{3.2}$$

Here  $u(n)$  is defined on  $n \in N_0^+$ ,  $\nabla u(n) = u(n) - u(n-1)$  is the backward difference of  $u(n)$ ,  $a$  and  $b$  are constants,  $f(n, r)$  is

A real valued function defined for  $(n, r) \in N_0^+ \times [0, \infty)$ . Equation (3.1) subject to (3.2) is discrete fractional boundary value problem (DFBVP).

Lemma3.1.

$$\text{For } n \in N_0^+, \sum_{j=0}^n D(n, \alpha, j) = \binom{n+\alpha}{n}.$$

Proof. Consider

$$\begin{aligned} \sum_{j=0}^n D(n, \alpha, j) &= \sum_{j=0}^n \binom{n-j+\alpha-1}{n-j} \\ &= \sum_{j=0}^n \frac{\Gamma(n-j+\alpha)}{\Gamma(\alpha)\Gamma(n-j+1)} \end{aligned}$$

$$= \frac{1}{\Gamma(n+1)} \sum_{j=0}^n \binom{n}{j} (1)_j (\alpha)_{n-j}$$

(using Def 2.2)

$$= \frac{1}{\Gamma(n+1)} (1+\alpha)_n$$

$$= \binom{n+\alpha}{n} \text{ (using Remark 1).}$$

Further it can be observed that  $D(N, \alpha, j)$  is increasing in  $j$  and  $D(N, \alpha, j) < 1$ , for  $j=0, 1, 2, \dots, N-1$  and when  $j=N$ ,  $D(N, \alpha, N)=1$ .

Theorem3.2. Suppose that there exists a constant  $M$  such that  $|f(j, u(j))| \leq M$ , then any solution  $u(n)$  of (3.1) satisfies

$$|u(n)| \leq |u(0)| + M \frac{\Gamma(n+\alpha)}{\Gamma(n)\Gamma(1+\alpha)}.$$

Proof: Let  $u(n)$  be a solution of (3.1).

From Remark 2, we have

$$u(n) = u(0) + \sum_{j=0}^{n-1} \binom{n-j+\alpha-2}{n-j-1} f(j, u(j))$$

Or

$$|u(n)| \leq |u(0)| + M \left| \sum_{j=0}^{n-1} \binom{n-j+\alpha-2}{n-j-1} \right|$$

$$= |u(0)| + M \left| \binom{n-1+\alpha}{n-1} \right|$$

(using Lemma 3.1)

$$= |u(0)| + M \frac{\Gamma(n+\alpha)}{\Gamma(n)\Gamma(1+\alpha)}.$$

Theorem3.3. Suppose that  $a > 0$  and  $b \geq 0$ , then the solution of the boundary value problem (3.1) and (3.2) satisfies

$$u(n) = -\frac{b}{a+b} \sum_{j=0}^N D(N, \alpha, j) f(j, u(j))$$

$$+ \sum_{j=0}^{n-1} D(n-1, \alpha, j) f(j, u(j))$$

Where  $D(n, \alpha, j) = \binom{n-j+\alpha-1}{n-j}$ .

Proof. Let  $u(n)$  be a solution of (3.1) which implies from Remark 2 that

$$u(n) = u(0) + \sum_{j=0}^{n-1} D(n-1, \alpha, j) f(j, u(j))$$

(3.3)

Substituting (3.3) in (3.2),

$$au(0) + b[u(0) + \sum_{j=0}^N D(N, \alpha, j) f(j, u(j))] = 0$$

Or

$$u(0) = \frac{-b}{a+b} \sum_{j=0}^N D(N, \alpha, j) f(j, u(j))$$

(3.4)

Substituting (3.4) in (3.3), we get

$$u(n) = -\frac{b}{a+b} \sum_{j=0}^N D(N, \alpha, j) f(j, u(j))$$

$$+ \sum_{j=0}^{n-1} D(n-1, \alpha, j) f(j, u(j))$$

(3.5)

Further, if  $|f(j, u(j))| \leq M$ , then

$$\begin{aligned}
 |u(n)| &\leq \frac{|b|}{|a+b|} \left| \sum_{j=0}^N D(N, \alpha, j) f(j, u(j)) \right| \\
 &+ \left| \sum_{j=0}^{n-1} D(n-1, \alpha, j) f(j, u(j)) \right| \\
 &\leq \frac{Mb}{a+b} \binom{N+\alpha}{N} + M \binom{N+\alpha}{N} \\
 &= M \frac{a+2b}{a+b} \binom{N+\alpha}{N}.
 \end{aligned}$$

Next we wish to establish the uniqueness of solutions to the boundary value problem (3.1) and (3.2). To achieve this, we define  $C[N_{n_0}^+, R]$ , the Banach space of all functions  $u(n)$  with norm defined by  $\|u\| = \max\{|u(n)|, n \in N_{n_0}^+\}$  and define  $F: C[N_{n_0}^+, R] \rightarrow C[N_{n_0}^+, R]$  by

$$\begin{aligned}
 Fu(n) &= -\frac{b}{a+b} \sum_{j=0}^N D(N, \alpha, j) f(j, u(j)) \\
 &+ \sum_{j=0}^{n-1} D(n-1, \alpha, j) f(j, u(j))
 \end{aligned}
 \tag{3.6}$$

**Theorem 3.4.** Suppose that the conditions of Theorem (3.3) hold good and there exists a constant  $L > 0$  such that

$$|f(t, u) - f(t, v)| \leq L|u - v|
 \tag{3.7}$$

For  $n \in N_{n_0}^+, u, v \in R$ . Further if

$\gamma = \frac{a+2b}{a+b} \binom{N+\alpha}{N} L < 1$ , then  $F$  has unique fixed point i.e.,

$$\begin{aligned}
 Fu(n) = u(n) &= -\frac{b}{a+b} \sum_{j=0}^N D(N, \alpha, j) f(j, u(j)) \\
 &+ \sum_{j=0}^{n-1} D(n-1, \alpha, j) f(j, u(j)).
 \end{aligned}$$

**Proof.** First we convert the problem into a fixed point problem by defining a map  $F: C[N_{n_0}^+, R] \rightarrow C[N_{n_0}^+, R]$  by (3.6).

For  $u, v \in R$ , consider

$$\begin{aligned}
 \|F_u - F_v\| &= \frac{b}{a+b} \sum_{j=0}^N D(N, \alpha, j) |f(j, u(j)) - f(j, v(j))| \\
 &+ \sum_{j=0}^{n-1} D(n-1, \alpha, j) |f(j, u(j)) - f(j, v(j))| \\
 &\leq L \left[ \frac{b}{a+b} \sum_{j=0}^N D(N, \alpha, j) + \sum_{j=0}^{n-1} D(n-1, \alpha, j) \right] |u - v| \\
 &\leq L \left[ \frac{b}{a+b} + 1 \right] \sum_{j=0}^N D(N, \alpha, j) |u - v| \\
 &\leq L \frac{a+2b}{a+b} \binom{N+\alpha}{N} |u - v|.
 \end{aligned}$$

In view of (3.8), we have  $\|F_u - F_v\| \leq \gamma \|u - v\|$  where  $\gamma < 1$ . Hence  $F$  has contraction map and hence by Banach fixed point theorem  $F$  has unique fixed point.

This result can be established using nonlinear contraction also.

Definition3.1. Let  $X$  be a Banach space and a map  $F: X \rightarrow X$  is said to be a nonlinear contraction if there exists a function  $\psi : R^+ \rightarrow R^+$  such that  $\psi(0) = 0$  and

$\psi(\eta) < \eta$  for  $\eta > 0$  satisfying the property

$$|f(t, u) - f(t, v)| \leq \psi(|u - v|) \quad (3.9)$$

Lemma3.5. [4] Let  $X$  be a Banach space and  $F: X \rightarrow X$  be a nonlinear contraction, then  $F$  has a unique point in  $X$ .

Theorem3.6. Let  $C [ N_{n_0}^+, R ]$  be a Banach space and a continuous function  $h: N_{n_0}^+ \rightarrow R^+$  be such that

$$|f(n, x) - f(n, y)| \leq h(n) \frac{|x - y|}{H^* + |x - y|}$$

$\forall x, y \geq 0, n \in N_{n_0}^+$  where

$$H^* = \frac{b}{a+b} \sum_{j=0}^N D(N, \alpha, j) h(j) + \sum_{j=0}^{n-1} D(n-1, \alpha, j) h(j).$$

Then boundary value problem (3.1) and (3.2) has unique solution.

Proof. Define the operator  $F: C [ N_{n_0}^+, R ] \rightarrow C [ N_{n_0}^+, R ]$  by (3.6) and consider the continuous and non-decreasing function  $\psi : R^+ \rightarrow R^+$  by

$$\psi(\rho) = \frac{H^* \rho}{H^* + \rho}$$

Clearly  $\psi(0) = 0$  and  $\rho > 0$ ,

$$\begin{aligned} \psi(\rho) - \rho &= \frac{H^* \rho}{H^* + \rho} - \rho > 0 \\ &= \frac{-\rho^2}{H^* + \rho} < 0 \text{ or } \psi(\rho) < \rho. \end{aligned}$$

Now consider

$$\begin{aligned} F_u - F_v &= -\frac{b}{a+b} \sum_{j=0}^N D(N, \alpha, j) f(j, u(j)) \\ &\quad + \frac{b}{a+b} \sum_{j=0}^N D(N, \alpha, j) f(j, v(j)) \\ &\quad + \sum_{j=0}^{n-1} D(n-1, \alpha, j) f(j, u(j)) \\ &\quad - \sum_{j=0}^{n-1} D(n-1, \alpha, j) f(j, v(j)) \end{aligned}$$

$$\begin{aligned} |F_u - F_v| &\leq \left| \frac{b}{a+b} \sum_{j=0}^N D(N, \alpha, j) [f(j, u(j)) - f(j, v(j))] \right| \\ &\quad + \left| \sum_{j=0}^{n-1} D(n-1, \alpha, j) [f(j, u(j)) - f(j, v(j))] \right| \\ &\leq \left[ \frac{b}{a+b} \sum_{j=0}^N D(N, \alpha, j) h_j + \sum_{j=0}^{n-1} D(n-1, \alpha, j) h_j \right] \\ &\quad \frac{|u - v|}{H^* + |u - v|} \\ &\leq \frac{H^* |u - v|}{H^* + |u - v|} \\ &= \psi(|u - v|). \end{aligned}$$

or  $\|F_u - F_v\| \leq \psi \|u - v\|$ . Hence  $F$  is a nonlinear contraction and by Lemma (3.5),  $F$  has a unique fixed point

which is a unique solution of the boundary value problem.

**4 EXAMPLE**

In this section, we illustrate the results established above by applying

them to two point boundary value problem. Consider the following discrete fractional boundary value problem.

Example 1: For  $n \in [0, 6]_{N_0^+}$ ,

$$\nabla^\alpha u(n+1) = \frac{|u(n) - 1|}{10[1 + |u(n) - 1|]} \tag{4.1}$$

$$u(0) + \frac{1}{2}u(6) = 0 \tag{4.2}$$

Set  $f(n, x) = \frac{|x(n)|}{10[1 + |x(n)|]}$ . Now consider

$$\begin{aligned} |f(n, u) - f(n, v)| &= \frac{1}{10} \left| \frac{|u(n)|}{[1 + |u(n)|]} - \frac{|v(n)|}{[1 + |v(n)|]} \right| \\ &\leq \frac{1}{10} \left| \frac{|u(n)| - |v(n)|}{[1 + |u(n)|][1 + |v(n)|]} \right| \\ &\leq \frac{1}{10} ||u(n)| - |v(n)|| \\ &\leq \frac{1}{10} |u(n) - v(n)|. \end{aligned}$$

Also, here  $L=1/10$ ,  $a=1$ ,  $b=1/2$  and hence for  $\alpha < 1$

$$\gamma = \frac{a + 2b}{a + b} \binom{N + \alpha}{N} = \frac{4}{30} \binom{6 + \alpha}{6}.$$

If  $\alpha = 1/2$ , then  $\gamma = 0.3910 < 1$ .

$$\text{If } \alpha = 0.9, \text{ then } \gamma = \frac{4}{30} \binom{6.9}{6} < \frac{4}{30} \cdot 7 = \frac{28}{30} < 1.$$

Hence the boundary value problem has unique solution.

Example 2: For  $n \in [0, 5]_{N_0^+}$ ,

$$\nabla^{1/2} u(n+1) = \frac{e^{-\sin^2 t} |u(n)|}{(n+3)^2 [1 + |u(n)|]} \tag{4.3}$$

$$u(0) + \frac{1}{3}u(5) = 0 \tag{4.4}$$

Here  $f(t, x) = \frac{e^{-\sin^2 t} |x(n)|}{(n+3)^2 [1 + |x(n)|]}$ ,  $\alpha = 1/2$

$$\begin{aligned} \text{Consider } |f(t, x(n)) - f(t, y(n))| &\leq \frac{e^{-\sin^2 t}}{(n+3)^2} \left[ \frac{|x(n)|}{1 + |x(n)|} - \frac{|y(n)|}{1 + |y(n)|} \right] \\ &\leq \frac{1}{(n+3)^2} \left| \frac{|x(n)| - |y(n)|}{[1 + |x(n)|][1 + |y(n)|]} \right| \\ &\leq \frac{1}{9} |x(n) - y(n)|. \end{aligned}$$

Here  $L=1/9$ ,  $a=1$  and  $b= 1/3$ , now

$$L \frac{a + 2b}{a + b} \binom{N + \alpha}{N} = \frac{5}{4} \cdot \frac{1}{9} \binom{5 + 1/2}{5}$$



$$\begin{aligned} &\leq \frac{5}{36} \cdot \frac{\Gamma(6+1/2)}{\Gamma(3/2)\Gamma(6)} \\ &\leq \frac{5}{36} \cdot \frac{\frac{11}{2} \cdot \frac{9}{2} \cdot \frac{7}{2} \cdot \frac{5}{2} \cdot \Gamma(3/2)}{\Gamma(3/2)\Gamma(6)} \\ &= 0.375976 < 1. \end{aligned}$$

Hence the boundary value problem has unique solution.

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